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Borel quantum kinematics of rank k on smooth manifolds

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Abstract. We propose a model for quantizing a non-relativistic physical system that admits a finite number of internal degrees of freedom. We assume that the configuration space of the given system carries the structure of a smooth manifold M. The kinematics of the given system is assumed to be describable by a flow model. Under certain technical restrictions momentum and position operators are seen to act on a Hilbert space of square-integrable sections of a Hermitian vector bundle over M. We determine their necessary shape up to isometric isomorphism and distinguish two types. In particular, we show that the number of inequivalent quantizations depends on the topology of the underlying configuration space.

1. Introduction

We describe a quantization rule for a non-relativistic physical system with a smooth finite-dimensional manifold M as configuration space so that finitely many internal degrees of freedom may be taken into account and which may possibly show how the coupling between internal space and configuration space reflects the topology of M. The rule is an extension [1–3, 15] of methods developed by Segal [16] and Mackey [12] based on the following concepts. The regions in configuration space where the system may be localized are modelled as Borel subsets of M. To mimic motion we use the flows φ^t_X of complete vector fields X of M with the complete vector field regarded as generalized momentum. We denote the collection of all Borel subsets of M by $\mathcal{B}(M)$ and the collection of all complete vector fields by $\mathcal{X}_c(M)$. We call the pair $(\mathcal{B}(M), \mathcal{X}_c(M))$ together with the flow model

$$B \longrightarrow \varphi^X_t(B) = \{\varphi^X_t(m) \mid m \in B\}$$

a *classical Borel kinematics*. We quantize it by fixing a Hilbert space $\mathcal{H}$ and two maps

$$\mathbf{E} : \mathcal{B}(M) \longrightarrow \mathcal{L}^+(\mathcal{H}) \qquad \mathbf{P} : \mathcal{X}_c(M) \longrightarrow \mathcal{L}_{sa}(\mathcal{H})\,.$$

Here, $\mathcal{L}^+(\mathcal{H})$ and $\mathcal{L}_{sa}(\mathcal{H})$ denote the set of positive operators on $\mathcal{H}$ and the set of self-adjoint operators on $\mathcal{H}$ respectively.

The map E has to respect the structure of the Borel σ-algebra $\mathcal{B}(M)$ and the probability interpretation of quantum mechanics. As a consequence, E is a positive-operator-valued measure for M on $\mathcal{H}$. We restrict E further by requiring it to be a projection-valued measure.

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The map $\mathbf{P}$ is to respect, whenever possible, the Lie algebra structure of $\mathcal{X}_c(M)$ and for each $X \in \mathcal{X}_c(M)$ the flow model is quantized via a one-parameter unitary shift-group $\{\mathbf{U}(\varphi_t^X) \mid t \in \mathbf{R}\}$ which satisfies the *imprimitivity condition*

$$\mathbf{U}(\varphi_t^X)\mathbf{E}(B)\mathbf{U}(\varphi_{-t}^X) = \mathbf{E}\left(\varphi_t^X(B)\right) .$$

Both maps $\mathbf{E}$ and $\mathbf{P}$ need to be specified further. $\mathbf{E}$ is unique up to isometric isomorphy, if we fix the multiplicity k of the projection-valued measure. Then the Hilbert space $\mathcal{H}$ can be modelled as the space of square-integrable $\mathbf{C}^k$-valued functions on M with respect to some measure ν and the projection-valued measure acts through the characteristic functions of the elements of $\mathcal{B}(M)$.

In accordance with 'classical' quantum theory, we wish to realize each $\mathbf{P}(X)$ as a differential operator. However, we have different possible choices for defining a domain of differentiability in $\mathcal{H}$, i.e. on the point set $M \times \mathbf{C}^k$.

For $k = 1$, Doebner and co-authors [3] showed that a generic choice is that of replacing the given Hilbert space by a Hilbert space of square-integrable sections of a line bundle, i.e. a $\mathbf{C}^1$-bundle, over M. In the case of a projection-valued measure of multiplicity k, the natural choice is to pass from the given Hilbert space of square-integrable $\mathbf{C}^k$-valued functions on M with respect to some measure ν to a Hilbert space of square-integrable sections of a Hermitian vector bundle of rank k over M with respect to some measure ν. One assumes furthermore a locality condition for $\mathbf{P}$ to ensure that $\mathbf{P}$ is given as a differential operator of finite order with respect to the mentioned differential structure.

We call the procedure which we described above *Borel quantization* and the resulting object a *Borel quantum kinematics of rank k over M*.

With these specializations one can hope to obtain a classification of all inequivalent triples $(\mathcal{H}, \mathbf{E}, \mathbf{P})$ up to isometric isomorphy. For $k = 1$ a complete classification is known: $\mathcal{H}$ may be thought of as a Hilbert space of sections of a Hermitian line bundle with flat connection and the equivalence classes of differentiable elementary quantum kinematics are isomorphic to $\pi_1^*(M) \times \mathbf{R}$, where $\pi_1^*(M)$ denotes the character group of the first fundamental group of M.

We will now consider the case where the projection-valued measure is of multiplicity k in detail and will focus on the question of the possible shapes of the map $\mathbf{P}$ and consequences thereof.

The paper is organized as follows: in section 2 we formally define a quantum kinematics of rank k on M and derive the shape of the Hilbert space we will work with. We then determine the necessary shape of the momentum operators under the technical requirement of differentiability in section 3 and as a direct consequence we give the standard form of a differentiable quantum kinematics of rank k. Finally we investigate the question of equivalence of differentiable quantum kinematics and as a special case generalize the result of [2] and [3] stated above.

2. Definitions and preliminary results

Let M be a smooth manifold of dimension n, and $(\mathcal{B}(M), \mathcal{X}_c(M))$ the corresponding classical Borel kinematics. Denote by Diff(M) the diffeomorphism group of M. A triple $(\mathcal{H}, \mathbf{E}, \mathbf{U})$ consisting of a separable Hilbert space $\mathcal{H}$, a representation $\mathbf{U}$ of Diff(M) on $\mathcal{H}$ and a projection-valued measure $\mathbf{E}$ for M on $\mathcal{H}$ is called a *quantization* of $(\mathcal{B}(M), \mathcal{X}_c(M))$, if for each $X \in \mathcal{X}_c(M)$ the one-parameter subgroup $\{\mathbf{U}(\varphi_t^X) \mid t \in \mathbf{R}\}$ of U(Diff(M)) is unitary and satisfies the imprimitivity condition

$$\mathbf{U}(\varphi_t^X)\mathbf{E}(B)\mathbf{U}(\varphi_{-t}^X) = \mathbf{E}\left(\varphi_t^X(B)\right)$$

for each $B \in \mathcal{B}(M)$.

The map **P** is obtained from **U** via differentiation, i.e. **P**(X) is the generator of the one-parameter subgroup generated by X (its uniqueness and existence are guaranteed by the Stones theorem). **P**(X) is essentially self-adjoint. Following the line of argument mentioned in the introduction and developed in [2, 4, 5] and [3] we require the map **P** in addition to reflect to some extent the properties the momentum map traditionally has. The map **P** is called a *partial Lie homomorphism*, if

(1) $\mathbf{P}(X + a\mathrm{Y}) = \overline{\mathbf{P}(X) + a\mathbf{P}(Y)}$ whenever $X + aY \in \mathcal{X}_c(M)$ for $X, Y \in \mathcal{X}_c(M)$ and $a \in \mathbf{R}$, and

(2) $\overline{[\mathbf{P}(X), \mathbf{P}(Y)]} = \imath\mathbf{P}([X, Y])$ whenever $[\mathrm{X}, Y] \in \mathcal{X}_c(M)$ for $X, Y \in \mathcal{X}_c(M)$.

P is called *local*, if

$$\left(\text{trace}\left(T_y \circ \mathrm{E}(B)\right) = 1 \wedge X|B = 0\right) \Longrightarrow (\mathbf{P}(\mathrm{X})y, y) = (\mathbf{P}(0)y, y)$$

for all $X \in \mathcal{X}_c(M), B \in \mathcal{B}(M), y \in \mathcal{D}(\mathbf{P}(X)) \cap \mathcal{D}(\mathbf{P}(0))$, with $T_y \in \mathcal{T}(\mathcal{H})$ denoting the pure state corresponding to y.

We are now prepared to define the object to be investigated.

Definition 2.1. A triple $(\mathcal{H}, \mathbf{E}, \mathbf{P})$ is called a *Borel quantum kinematics of rank k* on M, if

(1) for each $X \in \mathcal{X}_c(M)$, $\mathbf{P}(X)$ is the generator of a one-parameter unitary subrepresentation of a representation **U** of Diff(M) on $\mathcal{H}$ such that ($\mathcal{H}$, **E**, **U**) is a quantization of $(\mathcal{B}(M), \mathcal{X}_c(M))$,

(2) **E** is a projection-valued measure of multiplicity k for M on $\mathcal{H}$,

(3) the map **P** is a local, partial Lie-homomorphism, and

(4) there is a common dense domain $\mathcal{D}^\infty$ for all **E**(B) and all **P**(X).

To distinguish Borel quantum kinematics of rank k, we say that $(\mathcal{H}_1, \mathbf{E}_1, \mathbf{P}_1)$ and $(\mathcal{H}_2, \mathbf{E}_2, \mathbf{P}_2)$ are *equivalent*, if there exists an isometric isomorphism

$$\mathbf{U} : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$$

such that **U** intertwines $\mathbf{E}_1(B)$ and $\mathbf{E}_2(B)$ for all $B \in \mathcal{B}(M)$, as well as $\mathbf{P}_1(X)$ and $\mathbf{P}_2(X)$ for all $X \in \mathcal{X}_c(M)$.

We now turn to the construction of all possible Borel quantum kinematics of rank k. For the sake of brevity we will omit the prefix Borel. We start with **E**. Since **E** is a projection-valued measure, the main theorem of spectral multiplicity theory (see section 9.4 in [18]) ensures the existence of a sequence $\{\nu_n\}$ of mutually singular σ-finite Borel-measures on M such that the given Hilbert space $\mathcal{H}$ is isometrically isomorphic with $\mathcal{K} = \bigoplus_{n=1}^{\infty} L^2(M, \mathbf{C}^n, \nu_n)$ and **E** is equivalent to the projection-valued measure **P** on $\mathcal{K}$ given as

$$\mathbf{P}(B)(f_1, f_2, \ldots) = (\chi_B f_1, \chi_B f_2, \ldots).$$

Here χ_B denotes the characteristic function of the set B.

Moreover, since $\mathbf{E}$ is homogeneous of degree k, we have $\nu_n = 0$ for $n \neq k$ and so, with $\nu = \nu_k$, there is an isometric isomorphism

$$\mathbf{V} : \mathcal{H} \longrightarrow L^2(M, \mathbf{C}^k, \nu)$$

such that the projection-valued measure $\mathbf{E}$ may be realized as a family of multiplication operators on $L^2(M, \mathbf{C}^k, \nu)$:

$$\left(\mathbf{V}\mathbf{E}(B)\mathbf{V}^{-1}\psi\right)(m) = \chi_B(m)\psi(m)$$

for all $B \in \mathcal{B}(M)$ and all $\psi \in L^2(M, \mathbf{C}^k, \nu)$.

Condition (1) in definition 2.1 ensures that for each $X \in \mathcal{X}_c(M)$ the pair $(\mathbf{W}_X, \mathbf{E})$ where $\mathbf{W}_X(t) = \mathbf{U}(\varphi_t^X)$ for each $t \in \mathbf{R}$ is a system of imprimitivity based on M acting in $\mathcal{H}$ (here we regard M as an $\mathbf{R}$-space via the flow of the vector field X). It then follows from lemma 9.9 in [18] that the measure class of $\mathbf{E}$ is invariant under the action of $\mathbf{R}$ and that the measure ν is smooth (i.e. ν is locally equivalent to Lebesgue-measure on $\mathbf{R}^n$). We may then apply theorem 9.11 in [18] to conclude that for fixed $X \in \mathcal{X}_c(M)$, the image of the one-parameter subgroup $\{\mathbf{U}(\varphi_t^X) \mid t \in \mathbf{R}\}$ on $L^2(M, \mathbf{C}^k, \nu)$ is given by

$$\left(\mathbf{V}\mathbf{U}(\varphi_t^X)\mathbf{V}^{-1}\psi\right)(m) = \left(\rho_t^X(m)\right)^{1/2} \Pi^X(t, m)\psi\left(\varphi_{-t}^X(m)\right)$$

where ρ_t^X denotes a version of the Radon–Nikodym derivative of the shifted measure $\nu \circ \varphi_{-t}^X$ relative to ν and Π^X is a strict ($\mathbf{R}$, $M, \mathcal{U}(k)$)-cocycle relative to the measure class of $\mathbf{E}$.

Theorem 9.11 in [18] also associates to $(\mathbf{W}_X, \mathbf{E})$ the cohomology class of the cocycle Π^X and thus makes possible a classification of the individual systems of imprimitivity in terms of cohomology classes of ($\mathbf{R}$, $M, \mathcal{U}(k)$)-cocycles. Since we are primarily interested in the larger object, namely the quantum kinematics as a whole, we will not pursue this classification, valid for individual $X \in \mathcal{X}_c(M)$, in what follows.

The quantization of the classical Borel kinematics induces a map $\mathbf{Q}$ of the algebra $C^\infty(M, \mathbf{R})$ of smooth real-valued functions on M into $\mathcal{L}_{sa}(\mathcal{H})$ given by

$$\mathbf{Q}(f) = \int_M f \, \mathrm{d}\mathbf{E}$$

where $\int_M f \, \mathrm{d}\mathbf{E}$ denotes the unique operator satisfying

$$\left\langle \int_M f \, \mathrm{d}\mathbf{E}x, x \right\rangle = \int_M f \, \mathrm{d}\mu_x$$

with $\mu_x(B) = \langle \mathbf{E}(B)x, x\rangle$ for $x \in \mathcal{H}$. Here $\langle , \rangle$ denotes the inner product in $\mathcal{H}$. The operator $\mathbf{Q}(f)$ is often referred to as a position operator. $\mathbf{Q}(f)$ is realized as a multiplication operator on $L^2(M, \mathbf{C}^k, \nu)$

$$(\mathbf{V}\mathbf{Q}(f)\mathbf{V}^{-1}\psi)(m) = f(m)\psi(m)$$

for all $f \in C^\infty(M, \mathbf{R}), \psi \in L^2(M, \mathbf{C}^k, \nu)$ and $m \in M$. In this realization it is easy to establish the following proposition.

Proposition 2.2. Let $(\mathcal{H}, \mathbf{E}, \mathbf{P})$ be a quantum kinematics of rank k on M and let $\mathbf{Q}$ be defined as above. Then for every $X \in \mathcal{X}_c(M)$, for all $f, g \in C^\infty(M, \mathbf{R})$ and for all real a:

(1) $\|\mathbf{Q}(f)\| \leqslant \sup_{m \in M} \mid f(m) \mid$
(2) $\mathbf{Q}(f) = 0 \Longleftrightarrow f(m) = 0 \quad \forall m \in M$
(3) $\mathbf{Q}(f) + a\mathbf{Q}(g) = \mathbf{Q}(f + ag)$ on $\mathcal{D}(f) \cap \mathcal{D}(g)$
(4) $\mathbf{Q}(f) \cdot \mathbf{Q}(g) = \mathbf{Q}(fg)$ on $\mathcal{D}(fg)$
(5) $\mathbf{U}(\varphi_t^X) \circ \mathbf{Q}(f) \circ \mathbf{U}(\varphi_{-t}^X) = \mathbf{Q}(f \circ \varphi_{-t}^X)$
(6) $[\mathbf{P}(X), \mathbf{Q}(f)] = \mathbf{Q}(Xf)$.

We next specify the domain $\mathcal{D}^\infty$ from definition 2.1. We already mentioned in the introduction that we wish to realize $\mathbf{P}(X)$ as a differential operator which is plausible, since each vector field X is a differential operator relative to the differential structure of M. However, there is, up to the present, no notion of differentiability in $L^2(M, \mathbf{C}^k, \nu)$. We may identify $\mathbf{C}^k$-valued functions on M with sections of the trivial Hermitian vector bundle $(\xi_0 = (M \times \mathbf{C}^k, \pi_M, M, \mathbf{C}^k), \langle,\rangle_0)$ of rank k over M, the inner product $\langle,\rangle_0$ being induced from the usual inner product on $\mathbf{C}^k$ (see [14], chapter 2). We may thus identify $L^2(M, \mathbf{C}^k, \nu)$ and $L^2(\xi_0, \langle,\rangle_0, \nu)$, the Hilbert space of square-integrable sections of ξ_0. In this case we have equipped the point set $M \times \mathbf{C}^k$ with the differential structure inherited from the natural differential structures of M and $\mathbf{C}^k$. In general, there is more than one possible way in which to equip $M \times \mathbf{C}^k$ with a differential structure. In fact, one can show that given an arbitrary Hermitian vector bundle $(\xi = (\mathcal{E}, \pi, M, \mathbf{C}^k), \langle,\rangle)$ of rank k on M , there exists a differential structure $D = (\mathcal{T}, \mathcal{A})$ on $M \times \mathbf{C}^k$ consisting of a locally compact second countable Hausdorff topology $\mathcal{T}$ for $M \times \mathbf{C}^k$ and a maximal C^∞-atlas $\mathcal{A}$ of local charts compatible with $\mathcal{T}$ such that $(\xi_D = ((M \times \mathbf{C}^k, D), \pi_M, M, \mathbf{C}^k), \langle,\rangle_0)$ is diffeomorphic with $(\xi, \langle,\rangle)$ and the corresponding diffeomorphism is an isometry on each fibre. Moreover, the σ-algebras $\mathcal{B}(M \times \mathbf{C}^k, \mathcal{T})$ and $\mathcal{B}(M) \otimes \mathcal{B}(\mathbf{C}^k)$ are equal. (See theorem 3 in [3] for the proof in the case $k = 1$. The proof readily generalizes to the case $k > 1$ with the obvious substitutions.)

Given an arbitrary, not necessarily trivial, Hermitian k-vector bundle $(\xi, \langle,\rangle)$, the Hilbert space $L^2(\xi, \langle,\rangle, \nu)$ of ν-square-integrable sections of ξ contains a dense subspace of differentiable sections, namely set $\mathrm{Sec}_0^\infty(\xi)$ of compactly supported smooth sections of ξ (see 6.7 in [17] and lemma 5.1.1.10 in [19]). By the result quoted above, there is a differential structure D on $M \times \mathbf{C}^k$ such that $(\xi, \langle,\rangle)$ and $(\xi_D, \langle,\rangle_0)$ are diffeomorphic and so the corresponding Hilbert spaces are isometrically isomorphic. Moreover, since the σ-algebras $\mathcal{B}(M \times \mathbf{C}^k, \mathcal{T})$ and $\mathcal{B}(M) \otimes \mathcal{B}(\mathbf{C}^k)$ coincide, $\mathrm{Sec}_0^\infty(\xi_D)$ is dense in $L^2(\xi_0, \langle,\rangle_0, \nu)$, so that we may replace $L^2(\xi_D, \langle,\rangle_0, \nu)$ by $L^2(\xi_0, \langle,\rangle_0, \nu)$. Thus we may identify $L^2(\xi, \langle,\rangle, \nu)$ and $L^2(M, \mathbf{C}^k, \nu)$. The projection-valued measure $\mathbf{E}$ is again realized as a family of multiplication operators on $L^2(\xi, \langle,\rangle, \nu)$ (see [2, 14]). We may thus conclude that a replacement of $L^2(M, \mathbf{C}^k, \nu)$ by $L^2(\xi, \langle,\rangle, \nu)$ does not affect the realization of the projection-valued measure $\mathbf{E}$, while the existence of a natural region of differentiability may enable us to gain further insight into the structure of the map $\mathbf{P}$. In what follows we will therefore choose $\mathcal{H}$ to be $L^2(\xi, \langle,\rangle, \nu)$ for some Hermitian k-vector bundle $(\xi, \langle,\rangle)$ and we choose the subspace $\mathrm{Sec}_0^\infty(\xi)$ as our dense domain.

We call a quantum kinematics of rank k *differentiable*, if it is equivalent to a quantum kinematics $(\mathcal{H}, \mathbf{E}, \mathbf{P})$ with $\mathcal{H} = L^2(\xi, \langle,\rangle, \nu)$ for some smooth Hermitian vector bundle $(\xi, \langle,\rangle)$ of rank k over M, if $\mathrm{Sec}_0^\infty(\xi) = \mathcal{D}^\infty$ and if $\mathbf{P}(X)$ leaves $\mathrm{Sec}_0^\infty(\xi)$ invariant for each $X \in \mathcal{X}_c(M)$.

3. The shape of the momentum operator in the case of a differentiable quantum kinematics of rank k

Suppose that the quantum kinematics $(\mathcal{H}, \mathrm{E}, \mathbf{P})$ is differentiable, i.e. $\mathcal{H} = L^2(\xi, \langle,\rangle, \nu)$, where $(\xi, \langle,\rangle)$ is an arbitrary but fixed smooth Hermitian vector bundle of rank k over M, $(\mathrm{E}(B)\sigma)(m) = \chi_B(m)\sigma(m)$ for all $\sigma \in \mathcal{H}$ and all $m \in M$, and $\mathbf{P}(X)\mathrm{Sec}_0^\infty(\xi) \subseteq \mathrm{Sec}_0^\infty(\xi)$ for each $X \in \mathcal{X}_c(M)$.

We now determine the shape of $\mathbf{P}(X)$ on $\mathrm{Sec}_0^\infty(\xi)$. Let ∇ denote a connection on ξ compatible with the Hermitian metric on ξ (its existence is guaranteed by proposition 1.11 in [20]) and define for $X \in \mathcal{X}_c(M)$ the linear operator

$$\mathbf{P}^\nabla(X) : \mathrm{Sec}_0^\infty(\xi) \longrightarrow \mathrm{Sec}_0^\infty(\xi)$$

by

$$\mathbf{P}^\nabla(X)\sigma = \imath\nabla_X\sigma + \frac{\imath}{2}\mathrm{div}_\nu(X)\sigma \qquad \forall\sigma \in \mathrm{Sec}_0^\infty(\xi)\,.$$

We compare $\mathbf{P}$ and $\mathbf{P}^\nabla$. Set

$$\mathbf{A}^\nabla(X) = \mathbf{P}(X) - \mathbf{P}^\nabla(X)\,.$$

Then for arbitrary $f \in C^\infty(M, \mathbf{R})$ and for $\sigma \in \mathrm{Sec}_0^\infty(\xi)$ one has with proposition 2.2 (6.)

$$[\mathbf{A}^\nabla(X), \mathbf{Q}(f)]\sigma = 0\,.$$

$\mathbf{P}$ is local by definition and $\mathbf{P}^\nabla$ is local by construction. Therefore, $\mathbf{A}^\nabla$ is also local and we have

$$\mathrm{supp}\mathbf{A}^\nabla(X)\sigma \subset \mathrm{supp}\sigma.$$

Thus $\mathbf{A}^\nabla(X)$ is a linear differential operator of global order 0 on ξ in the sense of definition 6.1 in [10]. Therefore $\mathbf{A}^\nabla(X) \in C^\infty(M, \mathrm{Hom}(\xi))$, where $\mathrm{Hom}(\xi)$ denotes the homomorphism bundle of ξ.

We next determine properties of the map

$$\mathbf{A}^\nabla : \mathcal{X}_c(M) \longrightarrow C^\infty(M, \mathrm{Hom}(\xi))\,.$$

The operator $\mathbf{P}^\nabla(X)$ is easily seen to be symmetric on $\mathrm{Sec}_0^\infty(\xi)$ and $\mathbf{P}(X)$ is symmetric by definition. Thus $\mathbf{A}^\nabla(X)$ is symmetric. Since both $\mathbf{P}^\nabla$ and $\mathbf{P}$ are semi-linear, $\mathbf{A}^\nabla$ is also semi-linear.

If (U, h) is a local trivialization of ξ on U and F is a frame for ξ on U, then

$$(\mathbf{P}(X)\sigma)(F) = \imath\mathrm{d}_X\sigma(F) + \imath\theta(X, F)\sigma(F) + \frac{\imath}{2}\mathrm{div}_\nu(X)\sigma(F) + \mathbf{A}^\nabla(X, F)\sigma(F)\,.$$

Here d_X denotes the directional derivative in the direction X and $\theta(\cdot, F)$ denotes the connection matrix of ∇ in the frame F. Let $g \in C^\infty(\mathrm{U}, GL(k, \mathbf{C}))$ be a change of frame of ξ over U. Then because of

$$(\mathbf{P}(X)\sigma)(Fg) = g^{-1}(\mathbf{P}(X)\sigma)(F)$$

$\mathbf{A}^{\nabla}$ transforms under g like a section of $\mathrm{Hom}(\xi)$:

$$\mathbf{A}^{\nabla}(X, Fg) = g^{-1}\mathbf{A}^{\nabla}(X, F)g\,.$$

Therefore, $\mathbf{A}^{\nabla}$ is the restriction of a linear differential operator

$$\omega : \mathcal{X}(M) \longrightarrow \mathrm{Sec}^{\infty}(\mathrm{Hom}(\xi))\,.$$

As a consequence of the requirement that $\mathbf{P}$ be a partial Lie homomorphism it follows from

$$\mathbf{P}([X,Y])\sigma = \imath\nabla_{[X,Y]}\sigma + \frac{\imath}{2}\mathrm{div}_{\nu}([X,Y])\sigma + \omega([X,Y])\sigma$$

and

$$[\mathbf{P}(X), \mathbf{P}(Y)]\sigma = -[\nabla_X, \nabla_Y]\sigma - \tfrac{1}{2}(X\mathrm{div}_{\nu}(Y) - Y\mathrm{div}_{\nu}(X))\sigma + \cdots$$
$$\cdots + \imath(\nabla_X\omega(Y) - \nabla_y\omega(X))\sigma + [\omega(X), \omega(Y)]\sigma$$

for each $\sigma \in \mathrm{Sec}_0^{\infty}(\xi)$, that

$$\frac{1}{\imath}\Omega(X,Y) = (\mathrm{D}\omega)(X,Y) - \imath[\omega(X), \omega(Y)]$$

where Ω denotes the curvature of ∇ and D stands for the exterior covariant derivative with respect to ∇. Ω, regarded as a differential operator from $\mathcal{X}(M) \times \mathcal{X}(M)$ into $\mathrm{Sec}^{\infty}(\mathrm{Hom}(\xi))$, is of local order 0 for every local coordinate representation. By introducing local coordinates and writing ω as a differential operator of local order m, one finds via a cumbersome comparison of coefficients that there is a differential one-form ω_0 with values in $\mathrm{Hom}(\xi)$ and a self-adjoint section $\phi \in \mathrm{Sec}^{\infty}(\mathrm{Hom}(\xi))$ such that

$$\omega(X) = \omega_0(X) + \phi\mathrm{div}_{\nu}(X) \qquad \forall X \in \mathcal{X}(M)$$

i.e. ω is of at most order 1 (see [14, 2]). ϕ and ω_0 are related as follows. If $\hat{\nabla}$ denotes the connection on $\mathrm{Hom}(\xi)$ induced by ∇ then $\phi \in \mathrm{Sec}^{\infty}(\mathrm{Hom}(\xi))$ obeys

$$\hat{\nabla}_X\phi = \imath[\omega_0(X), \phi]$$

and ω_0 satifies

$$\frac{1}{\imath}\Omega(X,Y) = (D\omega_0)(X,Y) - \imath[\omega_0(X), \omega_0(Y)] \qquad \forall X, Y \in \mathcal{X}(M)\,.$$

The linear connection on ξ obtained by replacing ∇ by $\nabla - \imath\omega_0$ is again compatible with the Hermitian metric on ξ and, more importantly, is flat.

In summary, we have seen:

Proposition 3.1. Suppose that $(\mathcal{H}, \mathbf{E}, \mathbf{P})$ is a differentiable quantum kinematics of rank k on M with

$$\mathcal{H} = L^2(\xi, \langle,\rangle, \nu) \qquad \mathbf{E}(B)\sigma = \chi_B \cdot \sigma \quad \forall\sigma \in \mathcal{H} \quad \forall B \in \mathcal{B}(M)$$

where $(\xi, \langle,\rangle)$ denotes a Hermitian vector bundle of rank k. Then ξ admits a flat connection ∇ which is compatible with the Hermitian metric $\langle,\rangle$, and there is a $u(k)$-valued $\hat{\nabla}$-parallel ϕ in $\mathrm{Sec}^{\infty}(\mathrm{Hom}(\xi))$, i.e. $\hat{\nabla}_X\phi = 0 \quad \forall X \in \mathcal{X}_c(M)$, ($\hat{\nabla}$ denotes the connection on $\mathrm{Hom}(\xi)$ induced by ∇) such that

$$\mathbf{P}(X)\sigma = \imath\nabla_X\sigma + \left(\frac{\imath}{2}\,\mathrm{id}_{\mathrm{Sec}(\xi)} + \phi\right)\mathrm{div}_{\nu}(X)\sigma$$

for all $X \in \mathcal{X}_c(M)$ and for all $\sigma \in \mathrm{Sec}_0^{\infty}(\xi)$.

Proposition 3.1 is the cornerstone of any classification. It restricts the possible number of differentiable quantum kinematics of rank k on M, insofar as the corresponding Hermitian vector bundle on M must admit a flat connection and enables us to associate to any differentiable quantum kinematics of rank k on M a triple $(\nu, (\xi, \langle,\rangle, \nabla), \phi)$ consisting of a smooth Borel-measure ν on M, a flat Hermitian vector bundle over M and a $u(k)$-valued $\hat{\nabla}$-parallel section ϕ of $\mathrm{Hom}(\xi)$.

Above we have deduced the necessary shape of a differentiable quantum kinematics of rank k. Differentiable quantum kinematics of rank k exist for each smooth manifold M (see [14]): Given a smooth manifold M, simply choose a Hermitian vector bundle $(\xi, \langle,\rangle)$ over M of rank k with flat connection ∇. Let ν be a smooth Borel-measure on M and set $\phi = 0$. Then

$$\mathcal{H} = L^2(\xi, \langle,\rangle, \nu)$$

$$\mathbf{E}(B)\sigma = \chi_B \cdot \sigma \quad \forall \sigma \in \mathcal{H} \quad \forall B \in \mathcal{B}(M)$$

$$\mathbf{P}(X)\sigma = \imath \nabla_X \sigma + \frac{\imath}{2}\mathrm{div}_\nu(X)\sigma \quad \forall X \in \mathcal{X}_c(M) \quad \forall \sigma \in \mathrm{Sec}_0^\infty(\xi)$$

is a differentiable quantum kinematics of rank k on M. Here, we have chosen $\phi = 0$. (For a proof see [15] in the case of an orientable manifold and [14] otherwise.) Note that in integrated form the choice $c = 0$ corresponds to each cocycle Π^X being a co-boundary, and so the individual systems of imprimitivity are the Koopman systems of imprimitivity of [18].

4. Equivalence of differentiable quantum kinematics of rank k

We consider as a last topic the question of equivalence of two quantum kinematics of rank k. The following proposition gives the full answer:

Proposition 4.1. Let $(\mathcal{H}_1, \mathbf{E}_1, \mathbf{P}_1)$ and $(\mathcal{H}_2, \mathbf{E}_2, \mathbf{P}_2)$ be two differentiable quantum kinematics of rank k with corresponding triples $(\nu_i, (\xi_i, \langle,\rangle_i, \nabla^i), \phi_i)$, $i = 1,2$. $(\mathcal{H}_1, \mathbf{E}_1, \mathbf{P}_1)$ and $(\mathcal{H}_2, \mathbf{E}_2, \mathbf{P}_2)$ are equivalent, iff there exists a diffeomorphism $\Gamma : \mathcal{E}_1 \longrightarrow \mathcal{E}_2$ with

(1) $\Gamma|\pi_1^{-1}(m) : \pi_1^{-1}(m) \longrightarrow \pi_2^{-1}(m)$ is linear,
(2) Γ is isometric,
(3) $\Gamma \circ (\nabla_X^1(\Gamma^{-1} \circ \sigma)) = \nabla_X^2 \sigma \quad \forall X \in \mathcal{X}(M), \sigma \in \mathrm{Sec}^\infty(\xi_2)$,
(4) $\Gamma \circ \phi_1 \circ \Gamma^{-1} = \phi_2$.

Remark. We will call a diffeomorphism from $\mathcal{E}_1$ to $\mathcal{E}_2$ with properties (1), (2) and (3) an *isometric isomorphism that intertwines the connections.*

Proof. Suppose that $\Gamma : (\xi_1, \langle,\rangle_1, \nabla^1) \longrightarrow (\xi_2, \langle,\rangle_2, \nabla^2)$ is such that it satisfies (1)–(4). Let ρ denote a version of the Radon–Nikodym derivative of ν_1 with respect to ν_2. We define $\mathbf{U} : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ by

$$\mathbf{U}\sigma = \sqrt{\rho}\exp(-\imath\phi_2 \ln \rho)\Gamma\sigma$$

for $\sigma \in \mathcal{H}_1$. By the properties of Γ, $\mathbf{U}$ is an isometric isomorphism. If B is an arbitrary Borel subset of M, then for $\tau \in \mathcal{H}_2$ and $m \in M$,

$$\begin{aligned}
&\left(\left(\mathbf{U} \circ \mathbf{E}_1(B) \circ \mathbf{U}^{-1}\right) \tau\right)(m) \\
&\quad = \left(\sqrt{\rho} \exp(-\imath\phi_2 \ln \rho) \circ \Gamma \circ \mathbf{E}(B) \circ \Gamma^{-1} \circ \sqrt{\frac{1}{\rho}} \exp(\imath\phi_2 \ln \rho)\tau\right)(m) \\
&\quad = ((\exp(-\imath\phi_2 \ln \rho)\chi_B \exp(\imath\phi_2 \ln \rho))\tau)(m) \\
&\quad = \chi_B(m)\tau(m) = \mathbf{E}_2(B)\tau(m).
\end{aligned}$$

Thus $\mathbf{U}$ intertwines $\mathbf{E}_1(B)$ and $\mathbf{E}_2(B)$ for each $B \in \mathcal{B}(M)$.

For $X \in \mathcal{X}_c(M)$ and $\tau \in \mathrm{Sec}_0^\infty(\xi_2)$, straightforward calculation yields

$$\mathbf{U}\left(\nabla_X^1(\mathbf{U}^{-1}\tau)\right) = -\tfrac{1}{2}(X \ln \rho)\tau + \imath(X \ln \rho)\phi_2\tau + \nabla_X^2\tau$$

and consequently $\mathbf{U} \circ \mathbf{P}_1(X) \circ \mathbf{U}^{-1} = \mathbf{P}_2(X)$. Since $\mathrm{Sec}_0^\infty(\xi_2)$ is dense in $\mathcal{H}_2$, $\mathbf{P}_2$ and $\mathbf{P}_1$ are equivalent. Thus $(\mathcal{H}_1, \mathbf{E}_1, \mathbf{P}_1)$ and $(\mathcal{H}_2, \mathbf{E}_2, \mathbf{P}_2)$ are equivalent.

Now suppose that the given quantum kinematics are equivalent. We have $\mathcal{H}_i = L^2(\xi_i, \langle,\rangle_i, \nu_i)$, $i = 1, 2$. Since ν_1 and ν_2 are smooth Borel-measures, they belong to the same measure class. Let ρ denote a version of the Radon–Nikodym derivative of ν_1 with respect to ν_2 and define

$$\mathbf{V} : \mathcal{H}_1 \longrightarrow \mathcal{H} = L^2(\xi_1, \langle,\rangle_1, \nu_2)$$

by

$$\mathbf{V}\sigma = \exp(-\imath\phi_1 \ln \rho)\sqrt{\rho}\sigma \qquad \forall \sigma \in \mathcal{H}_1.$$

$\mathbf{V}$ is an isometric isomorphism and for $\sigma \in \mathcal{H}$ and $B \in \mathcal{B}(M)$

$$\left(\mathbf{V} \circ \mathbf{E}_1(B) \circ \mathbf{V}^{-1}\right)\sigma = \chi_B\sigma := \mathbf{E}(B)\sigma.$$

For $X \in \mathcal{X}_c(M)$ and $\sigma \in \mathrm{Sec}_0^\infty(\xi_1)$ we have

$$\mathbf{P}(\mathrm{X})\sigma := \left(\mathbf{V} \circ \mathbf{P}_1(\mathrm{X}) \circ \mathbf{V}^{-1}\right)\sigma = \imath\nabla_X^1\sigma + \left(\frac{\imath}{2}\,\mathrm{id}_{\mathrm{Sec}(\xi_1)} + \phi_1\right)\mathrm{div}_{\nu_2}(X)\sigma.$$

Moreover, $\mathbf{V}\mathrm{Sec}_0^\infty(\xi_1) = \mathrm{Sec}_0^\infty(\xi_1)$. It is obvious that $(\mathcal{H}, \mathbf{E}, \mathbf{P})$ is equivalent with $(\mathcal{H}_1, \mathbf{E}_1, \mathbf{P}_1)$ and by transitivity with $(\mathcal{H}_2, \mathbf{E}_2, \mathbf{P}_2)$. Thus there exists an isometric isomorphism $\mathbf{W} : \mathcal{H} \longrightarrow \mathcal{H}_2 = L^2(\xi_2, \langle,\rangle_2, \nu_2)$ such that

$$\begin{aligned}
\mathbf{W} \circ \mathbf{E}(B) \circ \mathbf{W}^{-1} &= \mathbf{E}_2(B) \qquad && \forall B \in \mathcal{B}(M) \\
\mathbf{W} \circ \mathbf{Q}(f) \circ \mathbf{W}^{-1} &= \mathbf{Q}_2(f) && \forall f \in C^\infty(M, \mathbf{R}) \\
\mathbf{W} \circ \mathbf{P}(X) \circ \mathbf{W}^{-1} &= \mathbf{P}_2(X) && \forall X \in \mathcal{X}_c(M).
\end{aligned}$$

Here, $\mathbf{Q}$ and $\mathbf{Q}_2$ denote the quantizations of $C^\infty(M, \mathbf{R})$ induced by $\mathbf{E}$ and $\mathbf{E}_2$ respectively. Since, furthermore, $\mathbf{W}$ maps polynomials in $\mathbf{Q}$ and polynomials in $\mathbf{P}$ into polynomials in $\mathbf{Q}_2$ and polynomials in $\mathbf{P}_2$, we have $\mathbf{W}\mathrm{Sec}_0^\infty(\xi_1) = \mathrm{Sec}_0^\infty(\xi_2)$. Since $\mathbf{W}$ intertwines $\mathbf{Q}$ and $\mathbf{Q}_2$, it follows that $\mathbf{W}$ is $C^\infty(M, \mathbf{C})$-linear. Thus $\mathbf{W}$ is induced by an isometric isomorphism Γ: $(\xi_1, \langle,\rangle_1) \longrightarrow (\xi_2, \langle,\rangle_2)$. If $X \in \mathcal{X}(M)$ and $\sigma \in \mathrm{Sec}_0^\infty(\xi_2)$, we have

$$\mathbf{W} \circ \mathbf{P}(X) \circ \mathbf{W}^{-1}\sigma = \imath(\Gamma \circ \nabla_X^1 \circ \Gamma^{-1})\sigma + \left(\frac{\imath}{2}\,\mathrm{id}_{\mathrm{Sec}(\xi_2)} + (\Gamma \circ \phi_1 \circ \Gamma^{-1})\right) \mathrm{div}_{\nu_2}(X)\sigma\,.$$

A comparison of this expression with $\mathbf{P}_2(X)$ shows that for $\mathbf{W}$ to intertwine $\mathbf{P}_2$ and $\mathbf{P}$ we need

$$\imath\left((\Gamma \circ \nabla_X^1 \circ \Gamma^{-1}) - \nabla_X^2\right)\sigma = \left(\phi_2 - (\Gamma \circ \phi_1 \circ \Gamma^{-1})\right) \mathrm{div}_{\nu_2}(X)\sigma.$$

The left-hand side of this equation is a differential operator of (at most) order 1 on $\mathrm{Sec}_0^\infty(\xi_2)$ while the right-hand side is of order 0. For fixed σ the left-hand side is $C^\infty(M, \mathbf{R})$-linear in the vector fields while due to the properties of div_{ν_2}, this is true for the right-hand side iff $\phi_2 = \Gamma \circ \phi_1 \circ \Gamma^{-1}$. So we have for all $X \in \mathcal{X}(M)$ and for all $\sigma \in \mathrm{Sec}_0^\infty(\xi_2)$

$$\left((\Gamma \circ \nabla_X^1 \circ \Gamma^{-1}) - \nabla_X^2\right)\sigma = 0$$

and so Γ is an isometric isomorphism that intertwines the connections. This concludes the proof of the proposition.

We briefly discuss the following special cases where only the momentum operators differ. The given differentiable quantum kinematics differ

(1) in the choice of flat connection and hence in the choice of the section ϕ,

(2) in the choice of the section ϕ only.

Proposition 4.2. Let $(\mathcal{H}, \mathbf{E}, \mathbf{P}_1)$ and $(\mathcal{H}, \mathbf{E}, \mathbf{P}_2)$ be two differentiable quantum kinematics of rank k with

$$\mathbf{P}_i(X)\sigma = \imath\nabla_X^i\sigma + \left(\frac{\imath}{2}\,\mathrm{id}_{\mathrm{Sec}(\xi)} + \phi_i\right)\mathrm{div}_\nu(X)\sigma$$

for all $X \in \mathcal{X}_c(M)$, $\sigma \in \mathrm{Sec}_0^\infty(\xi)$ and $i = 1, 2$. Then they are equivalent iff there exists a unitary section $v \in \mathrm{Sec}^\infty(\mathrm{Hom}(\xi))$ such that $\phi_2 = v \circ \phi_1 \circ v^{-1}$ and $\nabla^2 = \nabla^1 + v \circ \hat{\nabla}^1 v^{-1}$.

Outline of proof. Since the underlying Hilbert spaces are equal, the isometric isomorphism from proposition 4.1 is a unitary operator on $\mathcal{H}$ and so the isometric isomorphism on $(\xi, \langle,\rangle)$ is fibre-wise unitary. The condition on ∇^1 and ∇^2 then follows in a straightforward manner.

If we now assume ∇^1 and ∇^2 to be equal, then we have as an easy consequence of proposition 4.2:

Corollary 4.3. Let $(\mathcal{H}, \mathbf{E}, \mathbf{P}_1)$ and $(\mathcal{H}, \mathbf{E}, \mathbf{P}_2)$ be two differentiable quantum kinematics of rank k with

$$\mathbf{P}_i(X)\sigma = \imath\nabla_X\sigma + \left(\frac{\imath}{2}\,\mathrm{id}_{\mathrm{Sec}(\xi)} + \phi_i\right)\mathrm{div}_\nu(X)\sigma$$

for all $X \in \mathcal{X}_c(M)$, $\sigma \in \mathrm{Sec}_0^\infty(\xi)$ and $i = 1, 2$. Then the two quantum kinematics are equivalent iff there exists a unitary $\hat{\nabla}$-parallel section $v \in \mathrm{Sec}(\mathrm{Hom}(\xi))$ such that $\phi_2 = v \circ \phi_1 \circ v^{-1}$.

The proof follows from the observation that $\nabla^1 = \nabla^2$ iff $v \circ (\hat{\nabla}^1 v^{-1}) = 0$ iff $\hat{\nabla}^1 v = 0$ iff v is $\hat{\nabla}$-parallel.

Finally, we introduce a subclass of differentiable quantum kinematics of rank k: those for which $\phi = c \cdot \mathrm{id}_{\mathrm{Sec}(\xi)}$ for some real c. We will call these *differentiable k-quantum kinematics of type 0.* Condition 4 of proposition 4.1 now simplifies to $c_1 = c_2$ and we have:

Proposition 4.4. Let $(\mathcal{H}_1, \mathbf{E}_1, \mathbf{P}_1)$ and $(\mathcal{H}_2, \mathbf{E}_2, \mathbf{P}_2)$ be two differentiable quantum kinematics of rank k with corresponding triples $(\nu_i, (\xi_i, \langle,\rangle_i, \nabla^i), c_i)$, $i = 1,2$. $(\mathcal{H}_1, \mathbf{E}_1, \mathbf{P}_1)$ and $(\mathcal{H}_2, \mathbf{E}_2, \mathbf{P}_2)$ are equivalent if and only if $c_1 = c_2$ and there exists an isometric isomorphism $\Gamma : \mathcal{E}_1 \longrightarrow \mathcal{E}_2$ that intertwines the connections.

Proposition 4.4 enables us to construct equivalence classes of differentiable k-quantum kinematics of type 0. We have as a corollary:

Corollary 4.5. Denote by $[V_{\mathrm{flat}}^{\mathrm{herm}}(k)](M)$ the set of isomorphy classes of flat Hermitian vector bundles of rank k on M. Then there is a bijection between the set of equivalence classes of differentiable k-quantum kinematics of type 0 and $[V_{\mathrm{flat}}^{\mathrm{herm}}(k)](M) \times \mathbf{R}$.

We end our discussion with a complete classification of all differentiable k-quantum kinematics of type 0 on M: If η is a complex vector bundle on M of rank k, then the corresponding frame bundle, a GL(k,C) principal bundle on M whose transition functions coincide with those of η (see [20], p 66), admits a flat connection iff η admits a flat linear connection (by corollary 3.22 in [6] and 18.(2) in [8]). Thus there is a bijective correspondence between flat GL(k,C)-bundles and flat complex vector bundles. If furthermore η admits a Hermitian metric which the given flat connection is compatible with, then the corresponding principal bundle is reducible to a $\mathcal{U}(k)$-bundle with flat connection (see propositions III.1.5 and II.6.2 in [11]). Thus there is a bijection between flat $\mathcal{U}(k)$-principal bundles on M and flat Hermitian vector bundles on M of rank k. By lemma 1 in [13] a $\mathcal{U}(k)$-principal bundle admits a flat connection iff it is induced from the universal covering bundle of M by a homomorphism h from the fundamental group $\pi_1(M)$ into $\mathcal{U}(k)$. Therefore, there is a bijective correspondence between the set conj(Hom($\pi_1(M)$,$\mathcal{U}(k)$)) of conjugacy classes of homomorphisms $h : \pi_1(M) \longrightarrow \mathcal{U}(k)$ and $[V_{\mathrm{flat}}^{\mathrm{herm}}(k)](M)$. We may hence rephrase corollary 4.5 as follows:

Proposition 4.6. The set of equivalence classes of differentiable k-quantum kinematics of type 0 on M is isomorphic with the Cartesian product

$$\mathrm{conj}(\mathrm{Hom}(\pi_1(M),\mathcal{U}(k))) \times \mathbf{R}.$$

Differentiable quantum kinematics of rank k and type 0 are, as has been seen above, quite amenable to classification. They indeed yield the straightforward generalization of the elementary quantum kinematics which has been discussed in [2] and [3]. Those not of type 0 appear to be far less tractable. Here, we essentially require the existence of a *global* section of the homomorphism bundle of the given vector bundle which is parallel with respect to the given *flat* connection. The local existence of such solutions is guaranteed by the flatness of the connection on the vector bundle (see the Frobenius' theorem in [7]). A subclass of solutions that may be of interest is that of constant solutions not equal to $c \cdot \mathrm{id}_{\mathrm{Sec}(\xi)}$. Solutions of this form will certainly exist when the given vector bundle is trivial, in which case we may choose the connection matrix to be 0. Then we can expect solutions parameterized by vectors in $\mathbf{R}^k$, namely by the eigenvalues of the matrix ϕ.

5. Outlook

We have shown, as promised in the introduction, that the quantization of a non-relativistic system moving on a manifold M depends significantly on the topology of M. For a certain class of Borel quantum kinematics, those of type 0, a complete classification is obtained, depending on $\mathrm{Hom}(\pi_1(M), U(k))$ and on a real number c. This number may be interpreted as a new quantum number. The degrees of freedom in $\mathbf{C}^k$ are interpreted as internal. There is no coupling between internal and external degrees of freedom in the case of a type 0 Borel quantum kinematics.

We will present results and a physical interpretation for Borel quantum kinematics not of type 0 in a subsequent paper.

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